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Exact eigensystems for some matrices arising from discretizations

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ABSTRACT

It is known, for example, that the eigenvalues of the $N \times N$ matrix A, arising in the discretization of the wave equation, whose only nonzero entries are $A_{kk+1} = A_{k+1k} = -1, k = 1, \dots, N-1$, and $A_{kk} = 2, k = 1, \dots, N$, are $2\{1 - \cos[p\pi/(N+1)]\}$ with corresponding eigenvectors $v^{(p)}$ given by $v_k^{(p)} = \sin[pk\pi/(N+1)], p, k = 1, \dots, N$. We show by considering a simple finite difference approximation to the second derivative and using the summation formulae for sines and cosines that these and other similar formulae arise in a simple and unified way.

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1. Introduction

It is known that the eigenvalues of the matrix

$$\mathbf{N} = \mathbf{N}(e,f,g) = \begin{pmatrix} f & g & \cdot & \cdots & \cdot & \cdot & \cdot \\ e & f & g & \cdots & \cdot & \cdot & \cdot \\ 0 & e & f & g & \cdot & \cdot & \cdot \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \cdot & e & f & g & \cdot \\ \cdot & \cdot & \cdot & \cdots & e & f & g \\ \cdot & \cdot & \cdot & \cdots & \cdot & e & f \end{pmatrix} \tag{1}$$

are

$$\lambda^{(p)} = f + \sqrt{eg}\cos\frac{p\pi}{N+1}, \quad p = 1,2,\ldots,N \tag{2}$$

and the kth components of the corresponding eigenvectors (not normalized) are

$$v_k^{(p)} = \left(\frac{e}{g}\right)^{\frac{k}{2}} \sin\frac{pk\pi}{N+1}. \tag{3}$$

See [1, pp. 154–156], for a brief derivation. For early work on eigenvalues of the tridiagonal matrix please see [2,3, pp. 66–74], and [4, pp. 115–133] for more recent studies on this subject.

The special case

$$\mathbf{A} = \mathbf{N}(-1,2,-1) = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \tag{4}$$

is a finite difference approximation to (minus) the second derivative defined on an interval with Dirichlet boundary conditions at the end points. **A** arises naturally not only in solving certain simple differential and partial differential equations (see also [5–7]) but in other contexts, such as one-dimensional random walks with barriers [8, pp. 436–438], and similar path-count contexts in graph theory [9, pp. 213–214]. The derivations of formulas like (2) and (3) are quite straightforward, but we have not seen a unified treatment. The special case of matrix **A** is treated elegantly but in isolation by Smith, but we find that it is illuminating to see how several related matrices may be considered uniformly when treated in parallel with the eigenproblems for the second-order derivative with Dirichlet or Neumann boundary conditions.

We begin by considering in (5) and (6) the eigenproblem for $-\mathrm{d}^2/\mathrm{d}x^2$ on the interval $[a,b]$ with several types of boundary conditions at a and at b. The simplest finite-difference approximation to $-\mathrm{d}^2/\mathrm{d}x^2$ leads naturally to an exceedingly simple basic matrix eigenproblems where that matrices have first and last rows reflecting the choice of a and b and boundary conditions at those points, as will become clear.

It is found that the discretized eigenfunctions for these continuum problems obtained by sampling at equally spaced points are the components of the eigenvectors of the corresponding matrix equations while the corresponding eigenvalues for the discrete problem differ from the eigenvalues for the continuum problem by quantities of the second order in λh^2, where λ is the eigenvalue and h the step size in the difference scheme.

Finally we transform the simple matrices arising from $-\mathrm{d}^2/\mathrm{d}x^2$ to obtain more general forms related to $\mathbf{N}(e,f,g)$ of (1).

2. Formulation of the eigenproblem

On solving wave problems by separation of variables in Cartesian coordinates we are often led to eigenvalue problems of the following type, to find λ and $u(x)$ such that

$$\frac{d^2u}{dx^2} + \lambda u = 0 \tag{5}$$

with Dirichlet (D) or Neumann (N) boundary conditions (BCs) at a and b

$$\mathrm{D}_a\colon u(a) = 0 \text{ or } \mathrm{N}_a\colon u'(a) = 0 \quad \text{and} \quad \mathrm{D}_b\colon u(b) = 0 \text{ or } \mathrm{N}_b\colon u'(b) = 0. \tag{6}$$

We shall refer to the combinations of these BCs as $\mathrm{D}_a\mathrm{D}_b, \mathrm{D}_a\mathrm{N}_b, \mathrm{N}_a\mathrm{D}_b$ and $\mathrm{N}_a\mathrm{N}_b$.

On discretizing (5) we typically set $x = kh$, and $u_k = u(kh)$, where h is the step size and k is an integer, $k = 1, \ldots, N$. Then using a simple second-order accurate discretization of (5) we write the finite difference approximation to (5) as

$$\frac{u_{k-1} - 2u_k + u_{k+1}}{h^2} + \lambda_{\text{disc}} u_k = 0 \tag{7}$$

or equivalently

$$u_{k-1} + u_{k+1} = (2 - \lambda_{\text{disc}} h^2) u_k. \tag{8}$$

Here λ_{disc} is the discrete approximation to λ and, as we shall see, differs from it by an error of $O(\lambda h^2)$.

Since d^2/dx^2 is translation invariant and also invariant under the reflection $x \to -x$ we may satisfy the BCs (6) by supposing u to be continued as either an odd function where a Dirichlet BC applies, or as an even function where a Neumann BC applies. Then D_a and D_b may be replaced by conditions of odd symmetry, and N_a and N_b may be replaced by conditions of even symmetry. Let us take $a = h/2, b = (N + 1/2)h$, for instance, and continue the vector $\{u_k\}$ by the appropriate symmetries, allowing the subscript k to take on any integer value. The difference scheme (7) will then apply to the full range of k from 1 to N and we may write it in matrix form

$$\begin{pmatrix} 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ & & & 1 & 0 & 1 & \\ & & & & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \\ u_{N+1} \end{pmatrix} = \mu \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix}, \tag{9}$$

where in accordance with Eq. (8)

$$\mu = 2 - \lambda_{\text{disc}} h^2.$$

Notice that u_0 and u_{N+1} appear on the left of (9), and that the matrix on the left is not square but $N \times (N + 2)$.

Let us now restate (9), writing the appropriate oddness or evenness conditions at $h/2, (N + 1/2)h$ as

$$u_0 = \mp u_1 \quad \text{and} \quad u_{N+1} = \mp u_N, \tag{10}$$

which we use to eliminate u_0, u_{N+1} in favor of u_1, u_N. We then arrive at the following matrix eigenvalue problems:

$$\begin{pmatrix} \mp 1 & 1 & \cdot & \cdots & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & \cdots & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \cdot & 1 & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & 1 & \mp 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} = \mu \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} \tag{11}$$

for the BCs $\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}, \mathrm{D}_{h/2}\mathrm{N}_{(N+1/2)h}, \mathrm{N}_{h/2}\mathrm{D}_{(N+1/2)h}, \mathrm{N}_{h/2}\mathrm{N}_{(N+1/2)h}$. Here the upper left $\mp$ corresponds to $\mathrm{D}_{h/2}$ or $\mathrm{N}_{h/2}$, respectively, and the lower right $\mp$ corresponds to $\mathrm{D}_{(N+1/2)h}$ or $\mathrm{N}_{(N+1/2)h}$ respectively. In (11) the same vector appears on the left and on the right, and the matrix is square.

Similarly, if we set $a = 0$ or h and $b = (N+1)h$ or Nh, and then consider $\mathrm{D}_0\mathrm{D}_{(N+1)h}, \mathrm{D}_0\mathrm{N}_{Nh}, \mathrm{N}_h\mathrm{D}_{(N+1)h}, \mathrm{N}_h\mathrm{N}_{Nh}$ obtained by setting $u_0 = 0$ for D_0, $u_0 = u_2$ for N_h, $u_{N+1} = 0$ for $\mathrm{D}_{(N+1)h}$, $u_{N+1} = u_{N-1}$, for N_{Nh} and eliminate u_0, u_{N+1}, we obtain from (8) the following matrix eigenvalue problem:

$$\begin{pmatrix} 0 & m & \cdot & \cdots & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & \cdots & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \cdot & 1 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdots & 1 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdots & 0 & q & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} = \mu \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix}, \tag{12}$$

where $m = 1$ or 2 according as BCs D_0 or N_h apply, and $q = 1$ or 2 according as BCs $\mathrm{D}_{(N+1)h}$ or N_{Nh} apply.

Notice that the second through the $(N-1)$th rows of the matrix stay the same when the BCs are changed, but the first row changes with the choice of a and the BC at a while the Nth row changes with the choice of b and the BC at b. Moreover, only the first two elements of row 1 and the last two elements of the row N change.

Let us refer to these elements as l, m, q, r and write

$$\mathbf{M}(l,m,q,r) = \begin{pmatrix} l & m & \cdot & \cdots & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & \cdots & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \cdot & 1 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdots & 1 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdots & 0 & r & q \end{pmatrix}. \tag{13}$$

3. The continuum eigenvalue problem

Let us now return to the underlying ODE (5) and BCs (6). The eigenfunctions and eigenvalues for the four combinations of BCs at a and b are easily seen to be

$$\mathrm{D}_a\mathrm{D}_b\colon\ u^{(p)}_{\mathrm{D}_a\mathrm{D}_b}(x) = \sin\left[\beta^{(p)}_{\mathrm{D}_a\mathrm{D}_b}(x-a)\right],\quad \beta^{(p)}_{\mathrm{D}_a\mathrm{D}_b} = \frac{\pi p}{b-a},\quad \lambda^{(p)}_{\mathrm{D}_a\mathrm{D}_b} = \left(\beta^{(p)}_{\mathrm{D}_a\mathrm{D}_b}\right)^2, \tag{14}$$

$$\mathrm{D}_a\mathrm{N}_b\colon\ u^{(p)}_{\mathrm{D}_a\mathrm{N}_b}(x) = \sin\left[\beta^{(p)}_{\mathrm{D}_a\mathrm{N}_b}(x-a)\right],\quad \beta^{(p)}_{\mathrm{D}_a\mathrm{N}_b} = \frac{\pi\left(p-\frac{1}{2}\right)}{b-a},\quad \lambda^{(p)}_{\mathrm{D}_a\mathrm{N}_b} = \left(\beta^{(p)}_{\mathrm{D}_a\mathrm{N}_b}\right)^2, \tag{15}$$

$$\mathrm{N}_a\mathrm{D}_b\colon\ u^{(p)}_{\mathrm{N}_a\mathrm{D}_b}(x) = \cos\left[\beta^{(p)}_{\mathrm{N}_a\mathrm{D}_b}(x-a)\right],\quad \beta^{(p)}_{\mathrm{N}_a\mathrm{D}_b} = \frac{\pi\left(p-\frac{1}{2}\right)}{b-a},\quad \lambda^{(p)}_{\mathrm{N}_a\mathrm{D}_b} = \left(\beta^{(p)}_{\mathrm{N}_a\mathrm{D}_b}\right)^2, \tag{16}$$

$$\mathrm{N}_a\mathrm{N}_b\colon\ u^{(p)}_{\mathrm{N}_a\mathrm{N}_b}(x) = \cos\left[\beta^{(p)}_{\mathrm{N}_a\mathrm{N}_b}(x-a)\right],\quad \beta^{(p)}_{\mathrm{N}_a\mathrm{N}_b} = \frac{\pi(p-1)}{b-a},\quad \lambda^{(p)}_{\mathrm{N}_a\mathrm{N}_b} = \left(\beta^{(p)}_{\mathrm{N}_a\mathrm{N}_b}\right)^2. \tag{17}$$

Here $p = 1, \ldots, \infty$ labels the eigenvalues and eigenvectors.

4. Relation between the discrete and the continuum problem

It is an interesting fact that for these simple problems the discretized versions of the eigenfunctions are exact eigenvectors of the discrete (matrix) problems. One easily sees this and obtains explicit formulae by using the sum formulas for the sine and cosine. Thus

Using $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ we find that

$$\sin(A + B) + \sin(A - B) = 2 \cos B \sin A \tag{18}$$

and similarly using $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ we find that

$$\cos(A + B) + \cos(A - B) = 2 \cos B \cos A. \tag{19}$$

To illustrate the application of (18) and (19) we shall for example discretize $u^{(p)}_{\mathrm{D}_a\mathrm{D}_b}(x) = \sin(\beta^{(p)}_{\mathrm{D}_a\mathrm{D}_b}(x - a))$. Take $\beta^{(p)}_{\mathrm{D}_a\mathrm{D}_b} = \frac{\pi p}{b-a}, x = kh, a = \frac{1}{2}h, b = \left(N + \frac{1}{2}\right)h, b - a = Nh, A = \beta^{(p)}_{\mathrm{D}_a\mathrm{D}_b}(x - a) = \beta^{(p)}_{\mathrm{D}_a\mathrm{D}_b}(k - 1/2)h$, and $B = \beta^{(p)}_{\mathrm{D}_a\mathrm{D}_b}h$ in (14) to see that

$$\begin{aligned} &u^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}\left[(k - 1)h\right] + u^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}\left[(k + 1)h\right] \\ &\quad = 2 \cos\left(\beta^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}h\right) u^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}(kh), \end{aligned} \tag{20}$$

i.e.

$$u_{k-1} + u_{k+1} = 2 \cos\left(\beta^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}h\right) u_k, \tag{21}$$

where

$$u_k = u^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h},k} \equiv u^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}(kh) = \sin\left[\beta^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}(k - 1/2)h\right]. \tag{22}$$

This says that $\mathbf{u}^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}} = \{u^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h},k}\} = \{u^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}(kh)\}$, for $k = 1, \ldots, N, p = 1, \ldots, N$, form the pth eigenvector of the matrix with $\{l, m, q, r\} = \{-1, 1, 1, -1\}$ and belongs to the pth eigenvalue

$$\mu^{(p)}_{\mathrm{D}_h\mathrm{D}_{(N+1/2)h}} = 2 \cos\left(\beta^{(p)}_{\mathrm{D}_h\mathrm{D}_{(N+1/2)h}}h\right). \tag{23}$$

Notice that for $p = 1, \ldots, N$ the vector $\mathbf{u}^{(p)}_{\mathrm{D}_h\mathrm{D}_{(N+1/2)h}}$ obtained by discretizing (sampling) the eigenfunctions for the continuum problem are precisely the eigenvectors of the matrix problem obtained by discretizing the differential operator, but the eigenvalue for the difference operator in (7), namely $2[1 - \cos(\beta^{(p)}_{\mathrm{D}_h\mathrm{D}_{(N+1/2)h}}h)]/h^2$ is not precisely $\lambda^{(p)}_{\mathrm{D}_h\mathrm{D}_{(N+1/2)h}} = (\beta^{(p)}_{\mathrm{D}_h\mathrm{D}_{(N+1/2)h}})^2$ but only approximately equal to it as we see from the following brief calculation using the power series for the cosine function:

$$\begin{aligned} &\frac{2}{h^2}\left[1 - \cos\left(\beta^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}h\right)\right] \\ &\quad = \frac{2}{h^2}\left[\frac{1}{2!}\left(\beta^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}h\right)^2 - \frac{1}{4!}\left(\beta^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}h\right)^4 + \cdots\right] \\ &\quad = \left(\beta^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}\right)^2\left[1 - \frac{1}{12}\left(\beta^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}\right)^2 h^2 + \cdots\right] \\ &\quad = \lambda^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}\left[1 + O\left(\lambda^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}h^2\right)\right] \end{aligned} \tag{24}$$

for small $\lambda^{(p)}_{\mathrm{D}_{h/2}\mathrm{D}_{(N+1/2)h}}h^2$.

Thus the discretized eigenfunction (14) is an eigenvector of the discretized problem

$$\begin{pmatrix} -1 & 1 & \cdot & \cdots & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & \cdots & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \cdot & 1 & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & 1 & -1 \end{pmatrix} \begin{pmatrix} u^{(p)}_1 \\ u^{(p)}_2 \\ \vdots \\ u^{(p)}_{N-1} \\ u^{(p)}_N \end{pmatrix} = 2\cos(\beta^{(p)}h) \begin{pmatrix} u^{(p)}_1 \\ u^{(p)}_2 \\ \vdots \\ u^{(p)}_{N-1} \\ u^{(p)}_N \end{pmatrix}. \tag{25}$$

By similar considerations we may obtain the eigenvalues and corresponding eigenvectors for the matrices $\mathbf{M}(l, m, q, r)$ corresponding to the eigenvalue problems with all combinations of BCs considered

Table 1
The items in rows 2 through 17 give the sixteen particular values of the quantities listed in the first row. Each entry in the second through eighth column are the values taken on by the quantities listed in the first entry in its column corresponding to the assigned value of in the first entry of the row. The values of and in the last three entries of that row may then be used in (26) to form the eigenvalues and eigenvector entries of the matrix of (13).

$\{l,m,q,r\}$	$\frac{a}{h}$	$\frac{b}{h}$	$\overline{N}=\frac{b-a}{h}$	BCs	$\bar{p}$	$\bar{k}$	F
$-1,1,-1,1$	$\frac{1}{2}$	$N+\frac{1}{2}$	N	DD	p	$k-\frac{1}{2}$	sin
$-1,1,1,1$	$\frac{1}{2}$	$N+\frac{1}{2}$	N	DN	$p-\frac{1}{2}$	$k-\frac{1}{2}$	sin
$-1,1,0,1$	$\frac{1}{2}$	$N+1$	$N+\frac{1}{2}$	DD	p	$k-\frac{1}{2}$	sin
$-1,1,0,2$	$\frac{1}{2}$	N	$N-\frac{1}{2}$	DN	$p-\frac{1}{2}$	$k-\frac{1}{2}$	sin
$1,1,-1,1$	$\frac{1}{2}$	$N+\frac{1}{2}$	N	ND	$p-\frac{1}{2}$	$k-\frac{1}{2}$	cos
$1,1,1,1$	$\frac{1}{2}$	$N+\frac{1}{2}$	N	NN	$p-1$	$k-\frac{1}{2}$	cos
$1,1,0,1$	$\frac{1}{2}$	$N+1$	$N+\frac{1}{2}$	ND	$p-\frac{1}{2}$	$k-\frac{1}{2}$	cos
$1,1,0,2$	$\frac{1}{2}$	N	$N-\frac{1}{2}$	NN	$p-1$	$k-\frac{1}{2}$	cos
$0,1,-1,1$	0	$N+\frac{1}{2}$	$N+\frac{1}{2}$	DD	p	k	sin
$0,1,1,1$	0	$N+\frac{1}{2}$	$N+\frac{1}{2}$	DN	$p-\frac{1}{2}$	k	sin
$0,1,0,1$	0	$N+1$	$N+1$	DD	p	k	sin
$0,1,0,2$	0	N	N	DN	$p-\frac{1}{2}$	k	sin
$0,2,-1,1$	1	$N+\frac{1}{2}$	$N-\frac{1}{2}$	ND	$p-\frac{1}{2}$	$k-1$	cos
$0,2,1,1$	1	$N+\frac{1}{2}$	$N-\frac{1}{2}$	NN	$p-1$	$k-1$	cos
$0,2,0,1$	1	$N+1$	N	ND	$p-\frac{1}{2}$	$k-1$	cos
$0,2,0,2$	1	N	$N-1$	NN	$p-1$	$k-1$	cos

above. Thus, if l,m,q,r are chosen so that the ordered pairs $\{l,m\},\{q,r\}$ in (13) each take on one of the four values $\{-1,1\},\{1,1\},\{0,1\}$, or $\{0,2\}$ and we define $\bar{p},\bar{k},\overline{N}$ and function F according to Table 1.

Notice particularly that in Table 1 the pairs $\{l,m\}$ and $\{q,r\}$ may independently take on just one of the four values $\{-1,1\},\{1,1\},\{0,1\},\{0,2\}$. This leads to the sixteen combinations listed.

Then the pth eigenvalue $\lambda^{(p)}$ and the kth element $u_k^{(p)}$ of the pth eigenvector $\mathbf{u}^{(p)}$ of $\mathbf{M}(l,m,q,r)$ are

$$\lambda^{(p)}=2\cos\frac{\pi\bar{p}h}{\overline{N}} \quad\text{and}\quad u_k^{(p)}=F\left(\frac{\pi\bar{p}\bar{k}}{\overline{N}}\right). \tag{26}$$

5. Generalization

By transforming the matrix $\mathbf{M}(l,m,q,r)$ of (13) in several ways we can exactly solve the eigenproblems for related matrices of the form

$$\mathbf{N}(l,m,q,r;e,f,g)=\begin{pmatrix} f+\sqrt{eg}l & gm & \cdot & \cdots & \cdot & \cdot & \cdot \\ e & f & g & \cdots & \cdot & \cdot & \cdot \\ 0 & e & f & g & \cdot & \cdot & \cdot \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \cdot & e & f & g & 0 \\ \cdot & \cdot & \cdot & \cdots & e & f & g \\ \cdot & \cdot & \cdot & \cdots & 0 & er & f+\sqrt{eg}q \end{pmatrix}, \tag{27}$$

where $\{l,m,q,r\}$ take on the 16 values indicated in Table 1, and e,f,g are arbitrary real, except that e and g should be nonzero with the same sign.

We shall now demonstrate by a series of transformations that $\mathbf{N}$ of (27), with appropriate restrictions on l,m,q,r,e,f,g, may be reduced to the standard cases we have considered, namely $\mathbf{M}(l,m,q,r)=\mathbf{N}(l,m,q,r;1,0,1)$. Since these transformations do not change the values of l,m,q,r we shall leave these quantities understood and write $\mathbf{N}(e,f,g)$.

We shall be considering the eigenproblem

$$\mathbf{N}(e,f,g)\mathbf{v}(e,f,g)=\mu(e,f,g)\mathbf{v}(e,f,g) \tag{28}$$

and we start with the basic problem which we have already solved

$$\begin{aligned}\mathbf{M}(l,m,q,r)\mathbf{u}^{(p)}(1,0,1)&=\mathbf{N}(1,0,1)\mathbf{u}^{(p)}(1,0,1)=\mu^{(p)}(1,0,1)\mathbf{u}^{(p)}(1,0,1)\\ &=\lambda^{(p)}\mathbf{u}^{(p)}(1,0,1).\end{aligned} \tag{29}$$

See (25) and (26) above.

Let us premultiply (29) by $\mathbf{D} = \mathrm{diag}\left[(e/g)^{\frac{k}{2}}, k = 1, \ldots, N\right]$ and write

$$\mathbf{DND}^{-1}\mathbf{Du} = \lambda\mathbf{Du}, \tag{30}$$

i.e.

$$\mathbf{N}\left(\sqrt{e/g}, 0, \sqrt{g/e}\right)\mathbf{Du} = \lambda\mathbf{Du}, \tag{31}$$

where we have dropped the superscript $^{(p)}$.

Now multiply by $\sqrt{ge}$ to get

$$\mathbf{N}(e, 0, g)\mathbf{Du} = \sqrt{ge}\lambda\mathbf{Du} \tag{32}$$

and add $f\mathbf{I}$ to $\mathbf{N}(e, 0, g)$ to get

$$\mathbf{N}(e, f, g)\mathbf{Du} = \left(f + \sqrt{ge}\lambda\right)\mathbf{Du} \tag{33}$$

and so we see finally that

$$\mu(e, f, g) = f + \sqrt{ge}\lambda \tag{34}$$

and, referring back to (28),

$$\mathbf{v}(e, f, g) = \mathbf{Du}(1, 0, 1). \tag{35}$$

More explicitly the eigenvalue $\mu(e, f, g)$ and the components $v_k(e, f, g)$ of the eigenvector $\mathbf{v}(e, f, g)$ are

$$\begin{aligned} &\mu(e, f, g) = f + \sqrt{ge}\lambda^{(p)} \quad \text{and} \\ &v_k(e, f, g) = \left(\frac{e}{g}\right)^{\frac{k}{2}} u_k^{(p)}, \quad k = 1, \ldots, N, \end{aligned} \tag{36}$$

where $\lambda^{(p)}$ is the pth eigenvalue and $u_k^{(p)}$ the kth component of the pth eigenvector of $\mathbf{M}(l, m, q, r)$ for the particular values of l, m, q, r under consideration (see Table 1). This completes our discussion of the eigenproblem.

6. Conclusions

We have shown that the eigenvalues and eigenvectors of matrices of the form $\mathbf{N}(l, m, q, r; e, f, g)$ given in Eq. (27) have elementary closed-form expressions.

The expressions for these quantities may be deduced first by considering simple special cases which may be understood from their close analogy with corresponding eigenproblems for the second derivative operator on a finite interval with either Dirichlet or Neumann boundary conditions at the two end points.

Once these basic simple cases are understood one may then proceed to generalize the results by transforming the matrix using a similarity transformation by a scalar multiple of the identity, multiplication by a scalar, and addition of a scalar multiple of the identity, each of which has a very simple effect on the eigenvalues and eigenvectors.

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